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XVII. *On the Method of determining, from the real Probabilities of Life, the Values of contingent Reversions in which Three Lives are involved in the Survivorship.* By Mr. William Morgan, F. R. S.

Read May 26, 1791.

HAVING been encouraged to the further pursuit of the doctrine of survivorships by the very honourable manner in which my two former Papers on this subject were received by the Royal Society, I think it my duty to submit the result of my labours to their consideration. The solutions of some of the following problems might have been derived from those which I have already communicated; but the direct investigation of each separate problem being certainly more satisfactory, and the rules obtained by this means in general more simple, I have considered no problem as connected with another, except the relation between them either immediately arises from the solution, or is necessary to prove the truth of it. Being anxious to render myself as concise as possible, I have been minute only in the investigation of the first problem, and have done little more than state the contingencies which will determine the survivorship in the others. By the assistance, however, of these, and the operations which are detailed in my former Papers, the theorems which I have given may be deduced without much difficulty.

In

In order to prevent unnecessary repetitions, it may not be improper to begin with explaining the different symbols which are used in the following pages.

A, $\left\{ \begin{array}{l} \text{denote the value of an annuity on the respective lives} \\ \text{B,} \\ \text{C,} \end{array} \right\}$ of A, B, or C.

D, denotes the value of S on the contingency of C's surviving A (by my 2d prob. Vol. LXXVIII).

E, denotes the same value on the contingency of B's surviving A, found by the same problem.

F, denotes the value of an annuity on a life one year younger than B.

G, denotes the value of the absolute reversion of S after the death of A.

H, denotes the value of an annuity on a life one year younger than A.

K, denotes the value of an annuity on a life one year younger than C.

L, denotes the value of an annuity on the longest of the three lives of A, B, and C.

M, denotes the value of S by the first problem on the contingency that A's life shall be the *first* that fails.

N, denotes the value of an annuity on a life one year older than A.

P, denotes the value of an annuity on a life one year older than B.

R, denotes the value of S on the contingency of B's dying after A (by my 3d prob. Vol. LXXVIII).

S, denotes the given sum.

T, denotes the value of an annuity on a life one year older than C.

V, denotes

V , denotes the perpetuity.

W , denotes the value of S on the contingency of C 's dying after A (by my 3d prob. Vol. LXXVIII).

α and a denote the number of persons living in a table of observations at the ages of H and A .

s , t , u , w , &c. denote the number of persons living at the end of the 1st, 2d, 3d, &c. years from the age of A .

β and b , denote the number of persons living at the ages of F and B .

m , n , o , p , &c. denote the number of persons living at the end of the 1st, 2d, 3d, &c. years from the age of B .

κ and c denote the number of persons living at the ages of K and C .

d , e , f , g , &c. denote the number of persons living at the end of the 1st, 2d, 3d, &c. years from the age of C .

a' , a'' , a''' , &c. } denote the decrements of life at the end
 b' , b'' , b''' , &c. } of the 1st, 2d, 3d, &c. years from the
 c' , c'' , c''' , &c. } respective ages of A , B , and C .

r , denotes the value of £. 1 increased by its interest for a year.

The combinations of two or three of the several letters A , B , C , F , H , &c. denote the values of annuities on the joint continuance of two or three of those respective lives.

PROBLEM I.

To determine the value of a given sum, payable if A should be the *first* that fails of the three lives A , B , and C .

SOLUTION.

In order to receive the given sum in the first year, it is necessary that one or other of four events should happen. 1st,
 That

That all the three lives should fail, and that A should die first. 2dly, That B should die *after* A, and C live. 3dly, That C should die *after* A, and B live. 4thly, That A only should die, and B and C both live. These several contingencies being expressed by $\frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{3abc}$, $\frac{a' \cdot \overline{b-m} \cdot d}{2abc}$, $\frac{a' \cdot \overline{c-d} \cdot m}{2abc}$, and $\frac{a' \cdot dm}{abc}$,

respectively, their sum will be $= \frac{1}{abc} \times \frac{a'bc}{3} + \frac{a'ma}{3} + \frac{a'mc}{6} + \frac{a'bd}{6}$. In

the second and following years one or other of the same events must take place in order to receive the given sum; that is, they must either all three die, A dying first; or only A and B must die, A dying first; or only A and C must die, A dying first; or only A must die, and B and C both live. The different fractions expressing those four contingencies for the second year being $\frac{a'' \cdot \overline{m-n} \cdot \overline{d-e}}{3abc}$, $\frac{a'' \cdot \overline{m-n} \cdot e}{2abc}$, $\frac{a'' \cdot \overline{d-e} \cdot n}{2abc}$, and $\frac{a'' \cdot ne}{abc}$; for

the third year $\frac{a''' \cdot \overline{n-o} \cdot \overline{e-f}}{3abc}$, $\frac{a''' \cdot \overline{n-o} \cdot f}{2abc}$, $\frac{a''' \cdot \overline{e-f} \cdot o}{2abc}$, and $\frac{a''' \cdot of}{abc}$, and so on, the whole value of the given sum will be =

$$\begin{aligned} & \frac{S \cdot a'}{abc} \times \frac{bc}{3} + \frac{md}{3} + \frac{mc}{6} + \frac{bd}{6} + \frac{S \cdot a''}{abc^2} \times \frac{md}{3} + \frac{ne}{3} + \frac{nd}{6} + \frac{me}{6} + \frac{S \cdot a'''}{abc^3} \times \\ & \frac{ne}{3} + \frac{of}{3} + \frac{oe}{6} + \frac{nf}{6} + \&c. \text{ which are } = \frac{S}{3abc} \times \frac{a'bc}{r} + \frac{a'md}{r^2} + \frac{a''' \cdot en}{r^3} + \&c. + \\ & \frac{S}{6abc} \times \frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a''' \cdot oe}{r^3} + \&c. + \frac{S}{6abc} \times \frac{a'db}{r} + \frac{a''em}{r^2} + \frac{a''' \cdot fn}{r^3} + \&c. \\ & + \frac{S}{3abc} \times \frac{a'md}{r} + \frac{a'' \cdot en}{r^2} + \frac{a''' \cdot fo}{r^3} + \&c. \end{aligned}$$

From the demonstration of the problem in my last Paper * it appears, that the first of

these series is $= \frac{S}{3} \times \frac{\beta \cdot \overline{FK-ABK}}{bc} - \frac{S}{3r} \times \overline{BC-ABC}$; that the se-

cond series is $= \frac{S}{6} \times \frac{\alpha \cdot \overline{BK-ABK}}{c} - \frac{S}{6r} \times \frac{m \cdot \overline{PC-APC}}{b}$; that the

* See Phil. Trans. Vol. LXXIX.

third series is $= \frac{S}{6} \times \frac{\beta \cdot \overline{FC - AFC}}{b} - \frac{S}{6r} \times \frac{d \cdot \overline{BT - ABT}}{c}$; and that the

fourth series is $= \frac{S}{3} \times \overline{BC - ABC} - \frac{S}{3r} \times \frac{md \cdot \overline{PT - APT}}{bc}$. These several expressions being added together will be found $= S$ into

$$\frac{x}{3c} \times \frac{\beta \cdot \overline{FK - AFK}}{b} + \frac{\overline{EK - ABK}}{2} + \frac{\beta}{6b} \times \overline{FC - AFC} + \frac{-1}{3r} \times \overline{BC - ABC} \\ - \frac{m \cdot \overline{PC - APC}}{6br} - \frac{d}{3cr} \times \frac{\overline{BT - ABT}}{2} + \frac{m \cdot \overline{PT - APT}}{b}.$$

This theorem gives the exact value when either B or C are the oldest of the three lives; but when A is the oldest, it will be necessary to exchange the symbols a' , a'' , a''' , &c. for $a-s$, $s-t$, $t-u$, &c. and the symbols m , n , o , &c. for $c-c'$, $c-c'+c'$, $c-c'+c'+c''$, &c. In this case the value of the given sum for the first year will be found $= \frac{S}{abc}$ into

$$\frac{abc}{2} - \frac{bsc}{2} + \frac{amc}{2} - \frac{msc}{2} - \frac{abc'}{6} + \frac{bsc'}{6} - \frac{amc'}{3} + \frac{msc'}{3}; \text{ for the second year} =$$

$$\frac{S}{abc^2} \text{ into } \frac{msc}{2} - \frac{mtc}{2} + \frac{nsc}{2} - \frac{ntc}{2} - \frac{msc''}{6} + \frac{mtc''}{6} - \frac{nsc''}{3} + \frac{ntc''}{3} - \frac{msc'}{2} + \frac{mtc'}{2} -$$

$$\frac{nsc'}{2} + \frac{ntc'}{2}; \text{ for the third year} = \frac{S}{abc^3} \text{ into } \frac{ntc}{2} - \frac{nuc}{2} + \frac{otc}{2} + \frac{ouc}{2} - \frac{ntc'''}{6} +$$

$$\frac{nuc'''}{6} - \frac{otc'''}{3} + \frac{ouc'''}{3} - \frac{nt \cdot \overline{c' + c''}}{2} + \frac{nu \cdot \overline{c' + c''}}{2} - \frac{ot \cdot \overline{c' + c''}}{2} + \frac{ou \cdot \overline{c' + c''}}{2};$$

and so on for the remaining years of A's life. Hence the

$$\text{whole value of the given sum will be} = \frac{S}{2ab} \times \frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c.$$

$$- \frac{S}{2ab} \times \frac{bs}{r} + \frac{mt}{r^2} + \frac{nu}{r^3} + \&c. + \frac{S}{2ab} \times \frac{am}{r} + \frac{ns}{r^2} + \frac{ot}{r^3} + \&c. - \frac{S}{2ab} \times$$

$$\frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c. - \frac{S}{6abc} \times \frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c. - \frac{S}{2abc} \times$$

$$\frac{msc'}{r} + \frac{nt \cdot \overline{c' + c''}}{r^2} + \&c. + \frac{S}{6abc} \times \frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c. + \frac{S}{2abc} \times$$

mtc'

$$\frac{mtc'}{r} + \frac{nu \cdot c' + c''}{r^2} + \&c. - \frac{S}{3abc} \times \frac{amc'}{r} + \frac{ns \cdot c''}{r^2} + \frac{ot \cdot c'''}{r^3} + \&c. - \frac{S}{2abcr} \times$$

$$\frac{msc'}{r} + \frac{ot \cdot c' + c''}{r^2} + \&c. + \frac{S}{3abc} \times \frac{msc'}{r} + \frac{nt \cdot c''}{r^2} + \frac{ou \cdot c'''}{r^3} + \&c. \times \frac{S}{2abcr} \times$$

$$\frac{nt \cdot c'}{r} + \frac{ou \cdot c' + c''}{r^2} + \&c. \dots$$
 The first four series are respectively =

$$\frac{a\beta \cdot HF}{2ab} \text{ (or } \frac{1+AB}{2r}) - \frac{\beta \cdot AF}{2b} + \frac{a \cdot HB}{2a} - \frac{AB}{2} \dots$$
 The first term of the fifth series, or $\frac{abc'}{r}$, is = $\frac{a\beta}{abr} \times \frac{ab}{a\beta} - \frac{ab \cdot c - c'}{a\beta c}$, the second term or $\frac{msc''}{r^2}$, is = $\frac{a\beta}{abr^2} \times \frac{ms}{a\beta} - \frac{ms \cdot c - c' + c''}{a\beta c} - \frac{msc'}{r^2} \dots$ the third term, or $\frac{ntc'''}{r^3}$, is = $\frac{a\beta}{abr^3} \times \frac{nt}{a\beta} - \frac{nt \cdot c - c' + c'' + c'''}{a\beta c} - \frac{nt' \cdot c' + c''}{r^3}$. Therefore the sum of the fifth and sixth series is = $-\frac{a\beta \cdot HF - HFC}{6ab} \left(-\frac{msc'}{3abcr^2} - \frac{nt \cdot c' + c''}{3abcr^3} - \&c. = \right) - \frac{1}{3r} \times \overline{AB - ABC}$. In the same manner the seventh series may be found = $\frac{\beta}{6b} \times \overline{AF - AFC} - \frac{mtc'}{6abcr^2} - \frac{nu \cdot c + c''}{6abcr^3} - \&c.$; and consequently the sum of the seventh and eighth series is = $\frac{\beta \cdot \overline{AF - AFC}}{6b} + \frac{s \cdot \overline{BN - BNC}}{3ar}$. Again, the ninth and tenth series may be found = $-\frac{a \cdot \overline{HB - HBC}}{3a} \left(-\frac{msc'}{6abcr^2} - \frac{ot \cdot c' + c''}{6abcr^3} - \&c. = \right) - \frac{m \cdot \overline{AP - APC}}{6br}$; and the eleventh and twelfth or two last series, = $\frac{AB - ABC}{3} \left(+\frac{ntc'}{6abcr^2} + \frac{uo \cdot c' + c''}{6abcr^3} + \&c. = \right) + \frac{ms \cdot \overline{PN - PNC}}{6abr}$. If all these expressions be added together, we shall have the value of the given sum = S into $\frac{\beta}{3b} \times \frac{a \cdot \overline{HF} + \frac{1}{2}\overline{HFC}}{a} - \overline{AF} + \frac{1}{2}\overline{AFC}$

$$+ \frac{1}{6} \times \frac{a \cdot \overline{HB+2HBC}}{a} - \overline{AB+2ABC} + \frac{1}{3r} - \frac{s \cdot \overline{BN-BNC}}{a} - \overline{AB-ABC}$$

$$+ \frac{m}{6br} \times \frac{s \cdot \overline{PN-PNC}}{a} - \overline{AP-APC}.$$

It may be observed, that the first four series are also = E, and therefore if this value be substituted instead of $\frac{\alpha\beta \cdot \overline{HF}}{2ab}$, &c.

the general rule will become = E + S into $-\frac{\alpha}{3a} \times \overline{HB-HBC} + \frac{\beta}{2b}$

$$\times \overline{HF-HFC} + \frac{r-1 \cdot \overline{AB-ABC}}{3r} + \frac{\beta \cdot \overline{AF-AFC}}{6b} - \frac{m \cdot \overline{AP-APC}}{6br} + \frac{s}{3ar}$$

$$\times \overline{BN-BNC} + \frac{2b}{m} \times \overline{PN-PNC} \dots \dots \text{Supposing the three}$$

lives to be equal, the first of these rules will become = $\frac{1+CC}{2r} -$

$$\frac{xx \cdot \overline{KK-CKK}}{6 \cdot cc} - \frac{1}{3r} \times \overline{CC-CCC} - \frac{x \cdot \overline{CK-CCK}}{6c} + \frac{d \cdot \overline{CT-CCT}}{6cr} -$$

$$\frac{\overline{CC+2CCC}}{6} + \frac{dd \cdot \overline{TT-CTT}}{6 \cdot cc \cdot r}, \text{ and the second} = \frac{r-1 \cdot \overline{V-CC}}{2r} +$$

$$\frac{r-1 \cdot \overline{CC-CCC}}{3r} - \frac{x \cdot \overline{CK-CCK}}{6c} - \frac{xx \cdot \overline{KK-CKK}}{6cc} + \frac{d \cdot \overline{CT-CCT}}{6cr} +$$

$$\frac{dd \cdot \overline{TT-CTT}}{6ccr} : \text{ and also the rule denoting the value (when}$$

either B or C are the eldest of the three lives) will become =

$$\frac{r-1 \cdot \overline{CC-CCC}}{3r} + \frac{x}{3c} \times \overline{CK-CCK} + \frac{xx}{3 \cdot cc} \times \overline{KK-CKK} - \frac{d}{c} \times$$

$$\frac{\overline{CT-CCT}}{3r} - \frac{dd}{3ccr} \times \overline{TT-CTT}. \text{ Let this last expression be called}$$

Q and compared with the first of the preceding expressions. In

this case we shall have $\frac{1+CC}{2r} - \frac{\overline{CC+2CCC}}{6} - \frac{\overline{CC-CCC}}{3r} - \frac{Q}{2} +$

$$\frac{r-1 \cdot \overline{CC-CCC}}{6r} = Q, \text{ from which Q may be easily found} =$$

$$\frac{s}{3} \times \frac{r-1 \cdot \overline{V-CCC}}{r}. \text{ Again, let the same expression, denoted}$$

by

by Q , be compared with the second of the preceding expressions, and we shall then have $\frac{r-1 \cdot V-CC}{2r} - \frac{Q}{2} + \frac{r-1 \cdot CC-CCC}{2r}$

$= Q$, and consequently $Q = S \times \frac{r-1 \cdot V-CCC}{3r}$ as before *

Now it is well known, that when the lives are all equal, the value of the reversion must be *one third* the difference between the perpetuity and the three joint lives, and therefore a demonstration arises of the truth of the whole solution. As a still further proof of this, the foregoing theorem may be immediately deduced from the series themselves: thus, the value of the

given sum for the first year will in this case be $= \frac{S \cdot c^3 - d^3}{3c^3 \cdot r}$; for

the second year it will be $= \frac{S \cdot d^3 - e^3}{3c^3 r^2}$; for the third year it will be

$= \frac{S \cdot e^3 - f^3}{3c^3 r^3}$. Hence the value of the whole reversion will be $=$

$$\frac{S}{3} \times \frac{1}{r} - \frac{d^3}{c^3 r} - \frac{e^3}{c^3 r^2} - \&c. + \frac{S}{3} \times \frac{d^3}{c^3 r^2} + \frac{e^3}{c^3 r^3} + \frac{f^3}{c^3 r^4} + \&c. = \frac{S}{3} \times$$

$$\frac{1}{r} - CCC + \frac{CCC}{r} = \frac{S}{3} \times \frac{r-1 \cdot V-CCC}{r} \dots Q. E. D.$$

PROBLEM II.

To determine the value of a given sum, payable if A should be the *second* that fails of the three lives A , B , and C .

SOLUTION.

The sum S may be received in the first year, provided either

* This expression may also be obtained from either of the above general rules, independent of the two others, in like manner as in the solution of the fourth problem.

of three events should happen; 1st, if the three lives should become extinct, and A be the second that fails; 2dly, if A should die after B, and C live; and, 3dly, if A should die after C, and B live. But in the second and following years, the given sum may be received, provided either of seven events should happen. 1st, If the three lives should fail in the year, A's life having been the second that failed. 2dly, If A only should die in the year, B having died before the beginning and C lived to the end of it. 3dly, If A only should die in the year, C having died before the beginning and B lived to the end of it. 4thly, If A should die after B in the year and C live. 5thly, If A should die after C in the year and B live. 6thly, if A and C should both die in the year (A dying first) and B's life should have failed in one or other of the foregoing years; and, 7thly, if A and B should both die in the year (A dying first) and C's life should have failed in the foregoing years. From the several expressions denoting these contingencies the whole value of the reversion may be found=

$$\begin{aligned} & \frac{S}{2ac} \times \frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c. + \frac{S}{2ac} + \frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c. + \\ & \frac{S}{2ab} \times \frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a'''n}{r^3} + \&c. + \frac{S}{2ab} \times \frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c. - \\ & \frac{2S}{3abc} \times \frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c. - \frac{2.S}{3abc} \times \frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c. - \\ & \frac{S}{3abc} \times \frac{a'mc}{r} + \frac{a''na}{r^2} + \frac{a'''oe}{r^3} + \&c. - \frac{S}{3abc} \times \frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c. \end{aligned}$$

The two first of these series are = D, .. the two next = E ... and it appears from the solution of the preceding problem that the four remaining series express double the value of the sum S, depending on the contingency of A's dying first (when B or C are the oldest of the three lives) with a negative sign.

The

The general rule, therefore, in this case will become = $D + E - 2M$.

But when A is the oldest life, recourse must be had, as in the second part of the preceding problem, to different symbols, and the value of the reversion will then be found =

$$\begin{aligned} & \frac{S}{ab} \times \frac{-\frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c.}{r} + \frac{S}{ab} \times \frac{\frac{bs}{r} + \frac{mt}{r^2} + \frac{nu}{r^3} + \&c.}{r} + \\ & \frac{S}{3abc} \times \frac{\frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c.}{r} - \frac{S}{3abc} \times \frac{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c.}{r} + \\ & \frac{2S}{3abc} \times \frac{\frac{amc'}{r} + \frac{nsc''}{r^2} + \frac{otc'''}{r^3} + \&c.}{r} - \frac{2S}{3abc} \times \frac{\frac{msc'}{r} + \frac{ntc''}{r^2} + \frac{ouc'''}{r^3} + \&c.}{r} + \\ & \frac{S}{abc} \times \frac{\frac{msc'}{r} + \frac{nt \cdot c' + c''}{r^2} + \&c.}{r} - \frac{S}{abc} \times \frac{\frac{mtc'}{r} + \frac{nu \cdot c' + c''}{r^2} + \&c.}{r} + \\ & \frac{S}{abc} \times \frac{\frac{nsc'}{r} + \frac{ot \cdot c' + c''}{r^2} + \&c.}{r} - \frac{S}{abc} \times \frac{\frac{nt \cdot c'}{r} + \frac{ou \cdot c' + c''}{r^2} + \&c.}{r} + \\ & \frac{S}{2ab} \times \frac{\frac{a-s \cdot b}{r} + \frac{s-t \cdot m}{r^2} + \frac{t-u \cdot n}{r^3} + \&c.}{r} + \frac{S}{2ab} \times \frac{\frac{a-s \cdot m}{r} + \frac{s-t \cdot n}{r^2} + \&c.}{r} + \\ & \frac{S}{2ac} \times \frac{\frac{a-s \cdot c}{r} + \frac{s-t \cdot c - c'}{r^2} + \&c.}{r} + \frac{S}{2ac} \times \frac{\frac{a-s \cdot c - c'}{r} + \frac{s-t \cdot c - c' + c''}{r^2} + \&c.}{r} + \\ & \&c. - \frac{S}{ab} \times \frac{\frac{am}{r} + \frac{ns}{r^2} + \frac{ot}{r^3} + \&c.}{r} + \frac{S}{ab} \times \frac{\frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c.}{r}. \end{aligned}$$

The last two, and the first ten series are = $-2M$, the eleventh and twelfth series are = D , and the thirteenth and fourteenth series = E ; consequently the general rule becomes = $D + E - 2M$, as before.

Supposing the lives were all equal, the above expression would be = $\frac{S \cdot r - 1 \cdot V - CC}{r} - \frac{2 \cdot S}{3r} \times \frac{r - 1 \cdot V - CCC}{r} = \frac{S}{3} \times \frac{r - 1}{r} \times V - 3CC - 2CCC$, which is known from other principles* to be the true value, and therefore the investigation is right. As a further demonstration, however, it may not be improper to

* See Phil. Transf. Vol. LXXIX.

observe,

observe, that this rule is immediately obtained from the different fractions which express the several contingencies in each year. For in this case, the value of S for the first year becomes

$= \frac{S}{r} \times \frac{1}{3} - \frac{dd}{cc} + \frac{2d^3}{c^3}$, for the second year $= \frac{S}{r^2} \times \frac{dd}{cc} - \frac{ee}{cc} - \frac{2d^3}{c^3} + \frac{2e^3}{c^3}$

and so on for the other years. These series being added

$$\begin{aligned} \text{together will be found } &= \frac{S}{3} \times \frac{1}{r} + \frac{2d^3}{c^3 \cdot r} + \frac{2e^3}{c^3 \cdot r^2} + \&c. - \\ &\frac{S}{3} \times \frac{3dd}{ccr} + \frac{3ee}{cc \cdot r^2} + \&c. + \frac{S}{3r} \times \frac{3dd}{ccr} + \frac{3ee}{cc \cdot r^2} + \&c. - \frac{S}{3r} \times \frac{2d^3}{c^3 \cdot r^2} + \frac{2e^3}{c^3 \cdot r^2} \\ &+ \&c. = \frac{S}{3} \times \frac{r-1}{r} \times V - \frac{3CC-2CCC}{3} \dots Q. E. D. \end{aligned}$$

PROBLEM III.

To determine the value of a given sum payable on the death of A, if his life should be the *last* that fails of the three lives A, B, C.

SOLUTION.

The given sum can be received in the first year only upon the extinction of the three lives, A having died last. In the second and following years it may be received provided either of four events should happen: 1st, if all the three lives should fail in the year, A dying last; 2dly, if A should die *after* C in the year, B having died in either of the foregoing years; 3dly, if A should die *after* B in the year, C having died in either of the foregoing years; 4thly, if only A should die in the year, B and C having both died before the beginning of it. The value therefore of the reversion (when B or C are older than

$$\begin{aligned}
\text{than A) will be} &= \frac{S}{a} \times \frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c. + \frac{S}{3abc} \times \frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} \\
&+ \&c. + \frac{S}{3abc} \times \frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c. + \frac{S}{6abc} \times \frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} \\
&+ \&c. + \frac{S}{6abc} \times \frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c. - \frac{S}{2ac} \times \frac{a'c}{r} + \frac{a'd}{r^2} + \frac{a'''e}{r^3} + \&c. \\
&- \frac{S}{2ac} \times \frac{a'd}{r} + \frac{a'e}{r^2} + \frac{a'''f}{r^3} + \&c. - \frac{S}{2ab} \times \frac{a'b}{r} + \frac{a'm}{r^2} + \frac{a'''n}{r^3} + \&c. - \\
&\frac{S}{2ab} \times \frac{a'm}{r} + \frac{a'n}{r^2} + \frac{a'''o}{r^3} + \&c. = G + M - D + E.
\end{aligned}$$

When the life of A is the oldest of the three lives, the symbols being changed as in the two preceding problems, the value

$$\begin{aligned}
\text{of the reversion will become} &= -\frac{S}{6abc} \times \frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c. - \\
&\frac{S}{2abcr} \times \frac{msc'}{r} + \frac{nt \cdot c' + c''}{r^2} + \&c. + \frac{S}{6abc} \times \frac{bsc'}{r} + \frac{mic''}{r^2} + \frac{nuc'''}{r^3} + \&c. + \\
&\frac{S}{2abcr} \times \frac{mt \cdot c'}{r} + \frac{nu \cdot c' + c''}{r^2} + \&c. - \frac{S}{3abc} \times \frac{amc'}{r} + \frac{ns''c}{r^2} + \frac{ot \cdot c'''}{r^3} + \&c. - \\
&\frac{S}{2abcr} \times \frac{ns \cdot c'}{r} + \frac{ot \cdot c' + c''}{r^2} + \&c. + \frac{S}{3abc} \times \frac{msc'}{r} + \frac{nt \cdot c''}{r^2} + \frac{ou \cdot c'''}{r^3} + \&c. + \\
&\frac{S}{2abcr} \times \frac{nt \cdot c'}{r} + \frac{ou \cdot c' + c''}{r^2} + \&c. + \frac{S}{2ac} \times \frac{a-s \cdot c'}{r} + \frac{s-t \cdot c''}{r^2} + \frac{t-u \cdot c'''}{r^3} \\
&+ \&c. + \frac{S}{acr} \times \frac{s-t \cdot c'}{r} + \frac{t-u \cdot c' + c''}{r^2} + \&c.
\end{aligned}$$

By the second part of the solution of the first problem, the first eight series may be found = M - E; and the last two

series being easily resolved into $\frac{S}{a} \times \frac{a-s}{r} + \frac{s-t}{r^2} + \frac{t-u}{r^3} + \&c. -$

$$\begin{aligned}
&\frac{S}{2ac} \times \frac{a-s \cdot c - c'}{r} + \frac{s-t \cdot c - c' + c''}{r^2} + \&c. - \frac{S}{2ac} \times \frac{a-s \cdot c}{r} + \frac{s-t \cdot c - c'}{r^2} \\
&+ \&c. \text{ are} = G - D. \text{ Hence the general rule, as in the former} \\
&\text{case, becomes} = G + M - D + E.
\end{aligned}$$

When the lives are all equal, the above expression will be changed into $\frac{S}{r} \times \frac{r-1}{r} \times \frac{V-C}{V-C} + \frac{V-CCC}{3} - \frac{V-CC}{V-CC} = \frac{S \cdot r-1}{r} \times \frac{V-L}{3}$; or “*one-third* the value of the absolute reversion after “the extinction of the longest of the three lives.” This is known from self-evident principles to be the true value, and therefore the foregoing solution is correct. The same rule may also be obtained immediately from the series; for the value in this case for the first year will be $= \frac{S}{r} \times \frac{1}{3} - \frac{d}{c} + \frac{dd}{cc} - \frac{d^3}{3c^3}$, for the second year $= \frac{S}{r^2} \times \frac{d^3}{3c^3} - \frac{c^3}{3c^3} - \frac{dd}{cc} + \frac{a}{c} + \frac{ec}{cc} - \frac{e}{c}$, and so on for the remaining years. Hence the whole value of the reversion will be $= \frac{S}{3r} \times L + \frac{S}{3} \times \frac{1}{r} - L = \frac{r-1}{r} \cdot \frac{S}{3} \times \frac{V-L}{3} \dots \dots Q. E. D.$

PROBLEM IV.

To determine the value of a given sum payable on the extinction of the lives of A and B, should they be the *first* that fail of the three lives A, B, and C.

SOLUTION.

The given sum may be received in the first year, either on the extinction of the three lives, C having died last; or on the extinction of only the two lives A and B, C having survived the year. In the second and following years the given sum may be received, provided either of six events should happen. 1st, If the three lives should become extinct in the year, C having been the last that failed. 2dly, If A only should

should die in the year, B having died before the beginning, and C lived to the end of it. 3dly, If B only should die in the year, A having died before the beginning, and C lived to the end of it. 4thly, If A and B should both die in the year, and C survive it. 5thly, If C should die after A in the year, B having died in either of the foregoing years. 6thly, if C should die after B in the year, A having died in either of the foregoing years. The fractions denoting these several contingencies being added together will be found =

$$\begin{aligned} & \frac{S}{2ac} \times \frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c. + \frac{S}{2ac} \times \frac{ad}{r} + \frac{a'e}{r^2} + \frac{a'''f}{r^3} + \&c. - \\ & \frac{S}{6abc} \times \frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c. + \frac{S}{2abc} \times \frac{m'd \cdot a'}{r} + \frac{ne \cdot a' + a''}{r^2} + \&c. - \\ & \frac{S}{3abc} \times \frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c. - \frac{S}{2abc} \times \frac{nd \cdot a'}{r} + \frac{oe \cdot a' + a''}{r^2} + \&c. + \\ & \frac{S}{6abc} \times \frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''rf}{r^3} + \&c. + \frac{S}{2abc} \times \frac{me \cdot a'}{r} + \frac{nf \cdot a' + a''}{r^2} + \&c. - \\ & \frac{2S}{3abc} \times \frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c. - \frac{S}{2abc} \times \frac{ne \cdot a'}{r} + \frac{of \cdot a' + a''}{r^2} + \&c. \end{aligned}$$

The first two series are = D; the third and fourth series are =

$$\begin{aligned} & \frac{2}{3r} \times \overline{BC - ABC} - \frac{\beta \cdot r \cdot K - A \cdot K}{6 \cdot bc}; \text{ the fifth and sixth series are } = - \\ & \frac{\alpha \cdot BK - ABK}{3c} - \frac{m \cdot PC - APC}{6br}; \text{ the seventh and eighth series } = \\ & \frac{\beta \cdot FC - AFC}{6b} + \frac{d \cdot BT - ABT}{3cr}; \text{ and the ninth and tenth series } = \\ & \frac{md \cdot PT - APT}{6bcr} - \frac{2 \cdot BC - ABC}{3}. \end{aligned}$$

Hence the whole value of the reversion will be = $D - \frac{S \cdot \alpha}{3c} \times \frac{\beta \cdot FK - AFK}{2b} + \overline{BK - ABK} + \frac{S \cdot \beta}{6b} \times \overline{FC - AFC} - \frac{2 \cdot S \cdot r - I}{3r} \times \overline{BC - ABC} - \frac{S \cdot m}{6br} \times \overline{PC - APC} + \frac{S \cdot d}{3cr} \times \overline{BT - ABT} + \frac{m \cdot PT - APT}{2b}.$

The preceding theorem expresses the value of the given sum whether C be the oldest, or B one of the two lives to be survived, and will therefore be sufficient in all cases. In order, however, still more fully to prove this, let A be supposed the oldest life, and instead of $\overline{b-m}$, $\overline{m-n}$, $\overline{n-o}$, &c. and a' , a'' , a''' , &c. let b' , b'' , b''' , &c. and $\overline{a-s}$, $\overline{s-t}$, $\overline{t-u}$, &c. be substituted; then will the value of the reversion be found =

$$\begin{aligned} & \frac{S}{2bc} \times \frac{cb'}{r} + \frac{db''}{r^2} + \frac{eb'''}{r^3} + \&c. + \frac{S}{2bc} \times \frac{a'b'}{r} + \frac{a''b''}{r^2} + \frac{a'''b'''}{r^3} + \&c. - \\ & \frac{S}{6abc} \times \frac{acb'}{r} + \frac{dsb''}{r^2} + \frac{eib'''}{r^3} + \&c. + \frac{S}{2abc} \times \frac{db}{r} + \frac{t \cdot b' + b''}{r^2} + \&c. - \\ & \frac{S}{3abc} \times \frac{csb'}{r} + \frac{dtb''}{r^2} + \frac{eub'''}{r^3} + \&c. - \frac{S}{2abc} \times \frac{db'}{r} + \frac{eu \cdot b + b''}{r^2} + \&c. + \\ & \frac{S}{6abc} \times \frac{adb'}{r} + \frac{seb''}{r^2} + \frac{fib'''}{r^3} + \&c. + \frac{S}{2abc} \times \frac{seb'}{r} + \frac{tf \cdot b' + b''}{r^2} + \&c. - \\ & \frac{2S}{3abc} \times \frac{dsb'}{r} + \frac{et \cdot b''}{r^2} + \frac{fu \cdot b'''}{r^3} + \&c. - \frac{S}{2abc} \times \frac{et \cdot b'}{r} + \frac{fu \cdot b' + b''}{r^2} + \&c. \end{aligned}$$

Let Σ denote the value of S on the contingency of C's surviving B, and the general rule deduced from the preceding

series will become = $\Sigma - \frac{S \cdot x}{3} \times \frac{x \cdot \overline{HK} - \overline{HBK}}{2a} + \overline{AK} - \overline{ABK} +$
 $\frac{S \cdot x}{6a} \times \overline{HC} - \overline{HBC} - \frac{2S \cdot x - 1}{3r} \times \overline{AC} - \overline{ABC} - \frac{S \cdot s}{6ar} \times \overline{NC} - \overline{NBC} +$
 $\frac{S \cdot d}{3cr} \times \overline{AT} - \overline{ABT} + \frac{s \cdot \overline{NT} - \overline{NBT}}{2a}$, which appears to be exactly the same rule with the foregoing, if the symbols of A and B be only exchanged for each other.

If the three lives be of the same age, both those rules will severally become = S into $\frac{r-1}{2r} \times \overline{V} - \overline{CC} - \frac{xx}{6cc} \times \overline{KK} - \overline{CKK} -$
 $\frac{x}{6c} \times \overline{CK} - \overline{CCK} - \frac{2 \cdot r - 1}{3r} \times \overline{CC} - \overline{CCC} + \frac{d}{6cr} \times \overline{CT} - \overline{CCT} + \frac{dd}{6ccr} \times$
 $\overline{TT} -$

$\overline{TT-CTT}$. The two expressions $-\frac{x \cdot \overline{KK-CKK}}{6c} + \frac{d}{6cr} \times \overline{CT-CCT}$, by resolving them into their respective series, will be found $= \frac{1-CC}{6r} + \frac{d}{6cr} + \frac{de}{6c^2r^2} + \frac{ef}{6c^2r^3} + \frac{fg}{6c^2r^4} + \&c.$, and the two expressions $\frac{dd \cdot \overline{TT-CTT}}{6ccr} - \frac{x}{c} \times \overline{CK-CKK} = \frac{CC}{6r} - \frac{d}{6cr} - \frac{de}{6ccr^2} + \frac{ef}{6ccr^3} + \frac{fg}{6ccr^4} + \&c.$; hence the sum of these four expressions will be $= -\frac{r-1 \cdot \overline{V-CC}}{6r}$, and consequently the general rule in this case will be $= \frac{S \cdot \overline{r-1}}{3r} \times \overline{V-3CC-2CCC}$, which is known to denote the true value from other principles*.

As a further proof of the accuracy of the preceding investigation, it may not be improper to observe, that this rule may be immediately obtained from the different fractions which express the value of the given sum in each year. For in this case the value of S in the first year is $= \frac{S}{3} \times \frac{1}{r} - \frac{3d^1}{6cr} + \frac{2d^3}{c^3 \cdot r}$, in the second year $= \frac{S}{3} \times \frac{2c^3}{c^3r^2} - \frac{3ce}{ccr^2} - \frac{2d^3}{c^3r^2} + \frac{3dd}{ccr^2}$, and so on in the other years, which expressions may be easily found $= \frac{S \cdot \overline{r-1}}{3r} \times \overline{V-3CC-2CCC}$. Q. E. D.

PROBLEM V.

To find the value of a given sum, payable on the death of A, if his life should be the *first* or *second* that fails of the three lives A, B, and C.

* See my Paper in the Phil. Transf. Vol. LXXIX.

SOLUTION.

In the first year the payment of the given sum depends upon either of four events. 1st, That all the three lives shall become extinct, the life of A having been the first or second that failed. 2dly, That A and B shall both die, and C live to the end of the year. 3dly, That A and C shall both die, and B live to the end of the year. 4thly, That only A shall die, and B and C both live to the end of the year. In the second and following years the payment of the given sum will depend upon either of eight events happening. 1st, That all the three lives shall drop in the year, A having been the first or second that failed. 2dly, That C survives, and only A and B die in the year. 3dly, That B survives, and only A and C die in the year. 4thly, That both B and C survive, and A only dies in the year. 5thly, That A dies in the year, B having died before the beginning, and C lived to the end of it. 6thly, That A in like manner dies in the year, C having died before the beginning, and B lived to the end of it. 7thly, That C dies after A in the year, B's life having failed in either of the preceding years. 8thly, That B dies after A in the year, C's life having failed in either of the preceding years. The fractions denoting these

$$\begin{aligned}
 \text{several contingencies are} &= \frac{S}{2ac} \times \frac{a'c}{r} + \frac{a'' \cdot d}{r^2} + \frac{a''' \cdot e}{r^3} + \&c. + \\
 &\frac{S}{2ac} \times \frac{a'd}{r} + \frac{a'' \cdot e}{r^2} + \frac{a'''f}{r^3} + \&c. + \frac{S}{2ab} \times \frac{a'b}{r} + \frac{a'' \cdot m}{r^2} + \frac{a'''n}{r^3} + \&c. + \\
 &\frac{S}{2ab} \times \frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c. - \frac{S}{3abc} \times \frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c. - \\
 &\frac{S}{6abc} \times \frac{a'm}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c. - \frac{S}{6abc} \times \frac{a'ba}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c. - \\
 &\frac{S}{3abc} \times \frac{a'md}{r} + \frac{a''n}{r^2} + \frac{a'''of}{r^3} + \&c. = D + E - M.
 \end{aligned}$$

This

This general rule gives the true value whether the life of A be older or younger than both or either of the lives of B and C. When the three lives are of equal age, the value of S for the first year will be $= \frac{S}{r} \times \frac{2}{3} + \frac{d^3}{e^3} - \frac{dd}{cc}$, for the second year $= \frac{S}{r^2} \times \frac{e^3}{3e^3} - \frac{ee}{cc} - \frac{d^3}{3e^3} + \frac{dd}{cc}$, and so on for the other years. Hence the whole value in this case will be $= \frac{S \cdot r - 1}{3r} \times 2V - 3CC - CCC$, which expression may also be derived from the general rule just given above, or $D + E - M$.

The solution of this problem may also be obtained either from the first and second, or from the third problems. In the one case the value of S is evidently equal to the *sum* of the two values determined by the two first-mentioned problems, or $D + E - 2M + M = D + E - M$. And in the other case its value is equal to the *difference* between the absolute value of the reversion after A ($= G$) and its value depending upon the contingency that A shall be the *last* life that shall fail, which being $= G + M - \overline{D + E}$ by the third problem, it follows, that the general rule on this supposition will be also $= D + E - M$. Q. E. D.

PROBLEM VI.

To find the value of a given sum payable on the death of A, should his life be the *second* or *third* that shall fail of the three lives A, B, and C.

SOLUTION.

The payment of the given sum in the first year will depend upon the contingency of either of three events. 1st, That

all the three lives shall become extinct, A having been the second or third that has failed. 2dly, That A shall die after B, and C live to the end of the year. 3dly, That A shall die after C, and B live to the end of the year. In the second and following years the given sum will become payable, provided either of eight events should happen. 1st, If all the three lives should fail in the year, A having been the second or third that died. 2dly, If A should die after B in the year, and C live to the end of it. 3dly, If A should die after C in the year, and B live to the end of it. 4thly, If A and C should both die in the year, B having died before the beginning of it. 5thly, If A and B should both die in the year, C having died before the beginning of it. 6thly, If A only should die in the year, the lives of B and C having become extinct in either of the preceding years. 7thly, If A should die in the year, B having died before the beginning, and C lived to the end of it. 8thly, if A should die in the year, C having died before the beginning, and B lived to the end of it. Hence the whole value of the reversion will be found =

$$\begin{aligned} & \frac{S}{a} \times \frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c. - \frac{S}{3abc} \times \frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c. - \\ & \frac{S}{6abc} \times \frac{a'inc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c. - \frac{S}{6abt} \times \frac{a'bd}{r} + \frac{a'me}{r^2} + \frac{a'''nf}{r^3} + \&c. - \\ & \frac{S}{3abc} \times \frac{a'md}{r} + \frac{a'ne}{r^2} + \frac{a'''of}{r^3} + \&c. = G - M. \end{aligned}$$

This rule is correct in all cases; but when the three lives are equal it becomes more simple, and is = $\frac{S \cdot r - 1}{3r} \times 2V - 3C - CCC$; which expression may likewise be obtained immediately from the series given above.

The solution of this problem, like that of the foregoing one, may also be derived from the first three problems; for the value

value of S is either equal to the *difference* between the absolute value of the reversion after the death of A and its value depending on the contingency that A shall be the *first* that fails (found by Prob. 1.), or it is equal to the *sum* of the two values depending on the contingencies that A shall be the *second* or *third* that fails (found by Prob. 2. and 3.). In both cases the general rule is $= G - M$. Q. E. D.

PROBLEM VII.

To find the value of a given sum payable on the death of A , should his life be the *first* or *last* that fails of the three lives A , B , C .

SOLUTION.

In order to receive the given sum in the first year, it is necessary that either of four events should happen. 1st, That all the three lives should fail, A having been the first or third that died. 2dly, That B should die after A , and C live. 3dly, That C should die after A , and B live. 4thly, That A only should die, and B and C both live. In the second and following years the given sum may be received, provided either of seven events should happen. 1st, If the three lives should fail, A having been the first or last that died. 2dly, if B should die after A in the year, and C live to the end of it. 3dly, If C should die after A in the year, and B live to the end of it. 4thly, If A only should die in the year, and B and C both live to the end of it. 5thly, If A 's life should fail after that of B in the year, C 's life having failed before the beginning of it. 6thly, If A should fail after C in the year, B having failed before the beginning of it. 7thly, If A only

should die in the year, B and C having died in either of the preceding years. From the fractions denoting these several contingencies the whole value of the reversion will be found =

$$\begin{aligned} & \frac{S}{a} \times \frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c. - \frac{S}{2ac} \times \frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c. - \\ & \frac{S}{2ac} \times \frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c. - \frac{S}{2ab} \times \frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a'''n}{r^3} + \&c. - \\ & \frac{S}{2ab} \times \frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c. + \frac{2S}{3abc} \times \frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c. + \\ & \frac{S}{3abc} \times \frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c. + \frac{S}{3abc} \times \frac{a'b'd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c. + \\ & \frac{2S}{3abc} \times \frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c. = G - \overline{D} + E + 2M. \end{aligned}$$

This general theorem will give the exact value in all cases; but when the lives are equal, it is rendered more simple, by substituting the several values of G, D, E, and M, and will then become = $\frac{S \cdot r-1}{3r} \times 2V - 3C - 3CC + 2CCC$; which expression may also be derived in this particular case from the different series denoting the value of S in each year.

The solution of this, like those of the two preceding problems, may likewise be obtained by the assistance of the first three problems. For the value of this contingent reversion is either equal to the *sum* of the two values of S payable on the death of A, if his life should be the first, or if it should be the last that fails (found by Prob. 1. and 3.), or it is equal to the *difference* between the value of the absolute reversion after A's death, and the value of the contingent reversion after A's death, provided he should be the *second* that fails of the three lives (found by Prob. 2.). In both cases the general rule becomes = $G - \overline{D} + E + 2M$. Q. E. D.

PROBLEM VIII.

To find the value of a given sum payable on the death of A or B, should *either* of them be the *first* that shall fail of the three lives A, B, and C.

SOLUTION.

In each year the payment of the given sum will depend upon either of six events. 1st, If the three lives should fail in the year, A or B having died first. 2dly, if A and B should die in the year, and A live. 3dly, If C should die after A in the year, and B live. 4thly, If C should die after B in the year, and A live. 5thly, If A only should die in the year, and B and C both live. 6thly, If B only should die in the year, and A and C both live. The fractions denoting these

$$\begin{aligned} \text{several contingencies are} &= \frac{2S}{3abc} \times \frac{bca'}{r} + \frac{md \cdot a''}{r^2} + \frac{nea'''}{r^3} + \&c. - \\ &\frac{S}{6abc} \times \frac{mca'}{r} + \frac{nda''}{r^2} + \frac{oea'''}{r^3} + \&c. + \frac{S}{3abc} \times \frac{bda'}{r} + \frac{mea''}{r^2} + \frac{nf \cdot a'''}{r^3} + \&c. + \\ &\frac{S}{6abc} \times \frac{mda'}{a} + \frac{nea''}{r^2} + \frac{ofa'''}{r^3} + \&c. + \frac{S}{2bc} \times \frac{b-m \cdot c}{r} + \frac{m-n \cdot d}{r^2} + \frac{n-o \cdot e}{r^3} + \\ &\&c. + \frac{S}{2bc} \times \frac{b-m \cdot d}{r} + \frac{m-n \cdot e}{r^2} + \frac{n-o \cdot f}{r^3} + \&c. - \frac{S}{2abc} \times \frac{bca'}{r} + \frac{md \cdot a' + a''}{r^2} \\ &+ \&c. + \frac{S}{2abc} \times \frac{mca'}{r} + \frac{nd \cdot a' + a''}{r^2} + \&c. - \frac{S}{2abc} \times \frac{bda'}{r} + \frac{me \cdot a' + a''}{r^2} + \&c. \\ &+ \frac{S}{2abc} \times \frac{md \cdot a'}{r} + \frac{ne \cdot a' + a''}{r^2} + \&c. = S \text{ into } \frac{2}{3c} \times \frac{\beta \cdot \overline{FK} - A\overline{FK}}{2b} + \\ &\overline{BK} - A\overline{BK} - \frac{\beta \cdot \overline{FC} - A\overline{FC}}{6b} + \frac{2 \cdot \overline{r-1} \cdot \overline{BC} - A\overline{BC}}{3r} + \frac{m \cdot \overline{PC} - A\overline{PC}}{6br} - \frac{d}{3cr} \times \\ &\overline{BT} - A\overline{BT} + \frac{m \cdot \overline{PT} - A\overline{PT}}{2b} + \Sigma \quad (\Sigma \text{ denoting the value of } S \text{ on} \\ &\text{the contingency of C's surviving B, as in Prob. 4.)} \end{aligned}$$

The above rule gives the exact value when C is the oldest of the three lives. But if A be the oldest, the symbols must be changed as in some of the foregoing problems, and the value in this case will be expressed by the series

$$\begin{aligned} & \frac{2S}{3abc} \times \frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c. - \frac{S}{6abc} \times \frac{amc'}{r} + \frac{nsc''}{r^2} + \frac{otc'''}{r^3} + \&c. - \\ & \frac{S}{6abc} \times \frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c. - \frac{S}{3abc} \times \frac{msc'}{r} + \frac{ntc''}{r^2} + \frac{ouc'''}{r^3} + \&c. + \\ & \frac{S}{ab} \times \frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c. - \frac{S}{ab} \times \frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c. - \\ & \frac{S}{abc} \times \frac{abc'}{r} + \frac{ms \cdot c' + c''}{r^2} + \&c. + \frac{S}{abc} \times \frac{msc'}{r} + \frac{nt \cdot c' + c''}{r^2} + \&c. \end{aligned}$$

From these several series, the general rule expressing the value of the reversion will be found = S into $\frac{r-1 \cdot \overline{V-AB}}{r} - \frac{a}{3a} \times$

$$\begin{aligned} & \frac{\beta \cdot \overline{HF-HFC}}{b} + \frac{\overline{HB-HBC}}{2} - \frac{\beta \cdot \overline{AF-AFC}}{6b} + \frac{2 \cdot \overline{r-1 \cdot AB-ABC}}{3r} + \\ & \frac{m \cdot \overline{AP-APC}}{6br} + \frac{S}{3ar} \times \frac{\overline{BN-BNC}}{2} + \frac{m \cdot \overline{PN-PNC}}{b}. \end{aligned}$$

When the lives are all equal, the first rule becomes =

$$\begin{aligned} & \frac{r-1 \cdot \overline{V-CC}}{2r} + \frac{2 \cdot \overline{r-1 \cdot CC-CCC}}{3r} + \frac{x \cdot \overline{KK-CKK}}{6cc} + \frac{x \cdot \overline{CK-CCK}}{6c} - \\ & \frac{d \cdot \overline{CT-CCT}}{6cr} - \frac{dd \cdot \overline{TT-CTT}}{6ccr}, \text{ and the second rule} = \frac{r-1 \cdot \overline{V-CC}}{r} \\ & + \frac{2 \cdot \overline{r-1 \cdot CC-CCC}}{3r} - \frac{x \cdot \overline{KK-CKK}}{3cc} - \frac{x \cdot \overline{KC-CCK}}{3c} + \frac{d \cdot \overline{CT-CCT}}{3cr} \\ & + \frac{dd \cdot \overline{TT-CTT}}{3ccr}. \end{aligned}$$

In the one case the four last fractions are = $\frac{r-1 \cdot \overline{V-CC}}{6r}$; and in the other case those fractions are = $-\frac{r-1 \cdot \overline{V-CC}}{3r}$; therefore, in both cases the general rule becomes = $\frac{2 \cdot S \cdot r-1}{3r} \times \overline{V-CCC}$, which is known to be the true value

from

from self-evident principles. This expression may also be immediately derived from the different fractions which denote the value of S in each year. For in the first year these fractions will be reduced to $\frac{2S}{3} \times \frac{1}{r} - \frac{a^3}{c^3 r}$, in the second year to $\frac{2S}{3} \times \frac{a^3 - c^3}{c^3 r^2}$, in the third year to $\frac{2S}{3} \times \frac{c^3}{c^3 r^3} - \frac{f^3}{c^3 r^3}$, and so on in the other years. Hence the whole value of the reversion will, as above, be $= \frac{2 \cdot S \cdot r - 1}{3r} \times \sqrt{V - CCC}$. Q. E. D. *

PROBLEM IX.

To determine the value of a given sum payable on the death of A or B , should *either* of them be the *second* that fails of the three lives A , B , and C .

SOLUTION.

The payment of the given sum in the first year will depend upon either of four events happening. 1st, That the three lives should fail, A or B having been the second that failed. 2dly, That A should die after C , and B live. 3dly, That B should die after C , and A live. 4thly, That A and B should both die, and C live. In the second and following years the given sum will become payable, provided either of eleven events should happen. 1st, If the three lives should drop in the year, A or B having been the second that failed. 2dly, If A should die after C in the year, and B live. 3dly, If B should die after C in the year, and A live. 4thly, If A and B

* I do not know that any solution has been attempted before, either of this or of the two following problems.

should

should both die in the year, and C live. 5thly, if B only should die in the year, A having died before the beginning, and C lived to the end of it. 6thly, If A only should die in the year, B having died before the beginning, and C lived to the end of it. 7thly, If C should die after A in the year, B having died in either of the foregoing years. 8thly, If C should die after B in the year, A having died in either of the foregoing years. 9thly, If A and B should both die in the year, C having died in either of the preceding years. 10thly, If B only should die in the year, C having died before the beginning, and A lived to the end of it. And, lastly, if A only should die in the year, C having died before the beginning, and B lived to the end of it. The several fractions denoting these contingencies in each year being added together will

$$\begin{aligned}
 \text{become} = & \frac{S}{abc} \times \frac{bca'}{r} + \frac{md \cdot a' + a''}{r^2} + \&c. - \frac{S}{abc} \times \frac{mca'}{r} + \frac{nd \cdot a' + a''}{r^2} + \&c. \\
 & + \frac{S}{abc} \times \frac{bdda'}{r} + \frac{me \cdot a' + a''}{r^2} + \&c. - \frac{S}{abc} \times \frac{m'a'}{r} + \frac{ne \cdot a' + a''}{r^2} + \&c. - \\
 & \frac{4S}{3abc} \times \frac{bca'}{r} + \frac{mda''}{r^2} + \frac{nea'''}{r^3} + \&c. + \frac{S}{3abc} \times \frac{mca'}{r} + \frac{nda''}{r^2} + \frac{oea'''}{r^3} + \&c. - \\
 & \frac{2S}{3abc} \times \frac{bdda'}{r} + \frac{mea''}{r^2} + \frac{nfa'''}{r^3} + \&c. - \frac{S}{3abc} \times \frac{mda'}{r} + \frac{nea''}{r^2} + \frac{of \cdot a'''}{r^3} + \&c. - \\
 & \frac{S}{2bc} \times \frac{b-m \cdot c}{r} + \frac{m-n \cdot d}{r^2} + \frac{n-o \cdot e}{r^3} + \&c. - \frac{S}{2bc} \times \frac{b-m \cdot d}{r} + \frac{m-n \cdot e}{r^2} + \frac{n-o \cdot f}{r^3} \\
 & + \&c. + \frac{S}{ab} \times \frac{ma'}{r} + \frac{n \cdot a' + a''}{r^2} + \&c. - \frac{S}{ab} \times \frac{b'a'}{r} + \frac{m \cdot a' + a''}{r^2} + \&c. + \\
 & \frac{S}{ab} \times \frac{ba'}{r} + \frac{ma''}{r^2} + \frac{na'''}{r^3} + \&c. + \frac{S}{b} \times \frac{b-m \cdot c}{r} + \frac{m-n \cdot d}{r^2} + \frac{n-o \cdot e}{r^3} + \&c. + \\
 & \frac{S}{2ac} \times \frac{ca'}{r} + \frac{da''}{r^2} + \frac{ea'''}{r^3} + \&c. + \frac{S}{2ac} \times \frac{da'}{r} + \frac{ea''}{r^2} + \frac{fa'''}{r^3} + \&c. \quad \text{Let } Q
 \end{aligned}$$

denote the value of S by the first rule in the eighth problem, and the first ten series will be $= \Sigma - 2Q$. The four next series

are $= \frac{S \cdot r - 1 \cdot \overline{V - AB}}{r}$; and the two last series are $= D$; the whole value of the reversion, therefore (when C is the oldest of the three lives), is $= \frac{S \cdot r - 1 \cdot \overline{V - AB}}{r} + D + \Sigma - 2Q$.

When A is the oldest of the three lives, the different series, by changing the symbols, as in some of the foregoing problems, will

$$\begin{aligned} \text{become} &= \frac{2S}{abc} \times \frac{abc'}{r} + \frac{ms \cdot c' + c''}{r^2} + \&c. - \frac{2S}{abc} \times \frac{msc'}{r} + \frac{nt \cdot c' + c''}{r^2} + \&c. - \\ &\frac{4S}{3abc} \times \frac{abc'}{r} + \frac{msc''}{r^2} + \frac{nt \cdot c'''}{r^3} + \&c. + \frac{2S}{3abc} \times \frac{ms \cdot c'}{r} + \frac{nt \cdot c''}{r^2} + \frac{ou \cdot c'''}{r^3} + \&c. \\ &+ \frac{S}{3abc} \times \frac{amc'}{r} + \frac{nsc''}{r^2} + \frac{otc'''}{r^3} + \&c. + \frac{S}{3abc} \times \frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c. - \\ &\frac{S}{bc} \times \frac{bc'}{r} + \frac{m \cdot c' + c''}{r^2} + \&c. + \frac{S}{bc} \times \frac{mc'}{r} + \frac{n \cdot c' + c''}{r^2} + \&c. + \\ &\frac{S}{2bc} \times \frac{bc'}{r} + \frac{mc''}{r^2} + \frac{nc'''}{r^3} + \&c. - \frac{S}{2bc} \times \frac{mc'}{r} + \frac{nc''}{r^2} + \frac{oc'''}{r^3} + \&c. - \\ &\frac{S}{ac} \times \frac{ac'}{r} + \frac{s \cdot c' + c''}{r^2} + \&c. + \frac{S}{ac} \times \frac{sc'}{r} + \frac{t \cdot c' + c''}{r^2} + \&c. + \\ &\frac{S}{2ac} \times \frac{ac'}{r} + \frac{sc''}{r^2} + \frac{tc'''}{r^3} + \&c. - \frac{S}{2ac} \times \frac{sc'}{r} + \frac{tc''}{r^2} + \frac{uc'''}{r^3} + \&c. + \\ &\frac{S}{b} \times \frac{b-m}{r} + \frac{m-n}{r^2} + \frac{n-o}{r^3} + \&c. - \frac{S}{ab} \times \frac{b-m \cdot s}{r} + \frac{m-n \cdot t}{r^2} + \frac{n-o \cdot u}{r^3} + \&c. \\ &+ \frac{S}{abr} \times \frac{b-m \cdot s}{r} + \frac{m-n \cdot t}{r^2} + \&c. \end{aligned}$$

Let Q represent the value of S by the second rule in the eighth problem, Π and Δ the values of the same sum on the contingency of B's dying after C, and on the contingency of A's dying after C respectively *; then will the general rule in this case become $= \frac{S \cdot r - 1 \cdot 3\overline{V - B - A - AB}}{r}$

$- 2Q - \Pi + \Delta$, or, because $D + \Sigma$ are $= \frac{r - 1 \cdot 2\overline{V - B - A}}{r} - \Pi + \Delta$ †,

* See my 3d Prob. in Phil. Transf. Vol. LXXVIII.

† See my 2d and 3d Prob. Vol. LXXVIII.

it will be $= \frac{S \cdot \overline{r-1} \cdot \overline{V-AB}}{r} + D + \Sigma - 2Q$, as above.

When the lives are equal, the rule may be found either immediately from the series, or from the foregoing expressions, $= \frac{2S \cdot \overline{r-1}}{3r} \times V - 3CC - 2CCC$. Q. E. D.

PROBLEM X.

To find the value of a given sum payable on the decease of B or C, should *either* of them be the *last* that fails of the three lives A, B, and C.

SOLUTION.

The sum S can be received in the first year only on the extinction of the three lives, restrained to the contingency of A's life having been the first or second that failed. In the second and following years it may be received provided either of six events should happen. 1st, If the three lives should fail in the year, A having been the first or second that died. 2dly, If B and C should both die in the year, A having died before the beginning of it. 3dly, If C only should die in the year, A and B having died in either of the preceding years. 4thly, If B only should die in the year, A and C having died in either of the preceding years. 5thly, If B should die after A in the year, C having died before the beginning of it. 6thly, If C should die after A in the year, B having died before the beginning of it. From the fractions expressing these several contingencies, the whole value of the reversion will be found =

$$\frac{2S}{3abc} \times \frac{mda'}{r} + \frac{ne \cdot a' + a''}{r^2} + \&c. - \frac{S}{3abc} \times \frac{bca'}{r} + \frac{md \cdot a' + a''}{r^2} + \&c. -$$

$$\begin{aligned} & \frac{S}{6abc} \times \frac{mca'}{r} + \frac{nda''}{r^2} + \frac{oea'''}{r^3} + \&c. - \frac{S}{6abc} \times \frac{bda'}{r} + \frac{mea''}{r^2} + \frac{nf.a'''}{r^3} + \&c. + \\ & \frac{S}{2ab} \times \frac{ba'}{r} + \frac{m.a'+a''}{r^2} + \frac{n.a'+a''+a'''}{r^3} + \&c. - \frac{S}{2ab} \times \frac{ma'}{r} + \frac{n.a'+a''}{r^2} + \\ & \&c. + \frac{S}{2ac} \times \frac{ca'}{r} + \frac{d.a'+a''}{r^2} + \&c. - \frac{S}{2ac} \times \frac{da'}{r} + \frac{e.a'+a''}{r^2} + \&c. - \\ & \frac{2S}{3abc} \times \frac{mda'}{r} + \frac{ne.a'+a''}{r^2} + \&c. + \frac{S}{3abc} \times \frac{nea'}{r} + \frac{of.a'+a''}{r^2} + \&c. + \\ & \frac{S}{2abr} \times \frac{ma'}{r} + \frac{n.a'+a''}{r^2} + \&c. - \frac{S}{2abr} \times \frac{na'}{r} + \frac{o.a'+a''}{r^2} + \&c. + \\ & \frac{S}{2acr} \times \frac{d.a'}{r} + \frac{e.a'+a''}{r^2} + \&c. - \frac{S}{2acr} \times \frac{ea'}{r} + \frac{f.a'+a''}{r^2} + \&c., \text{ and} \end{aligned}$$

therefore the general rule (when either B or C are the oldest of the three lives) will be = S into $\frac{2.r-1.BC-ABC}{3r} - \frac{x}{3c} \times$

$$\begin{aligned} & \frac{\beta.FK-AFK}{b} + \frac{BK-ABK}{2} \times \frac{m}{3br} \times \frac{PC-APC}{2} + \frac{d.PT-AT}{c} - \frac{\beta.FC-AFC}{6b} \\ & + \frac{d.BT-ABT}{6cr} + \frac{\beta.F-AF}{2b} - \frac{r-1.B-AB}{2r} - \frac{m.P-AP}{2br} + \frac{x.K-AK}{2c} - \\ & \frac{r-1.C-AC}{2r} - \frac{d.T-AT}{2cr}; \text{ or (since the first five expressions are =} \\ & \frac{r-1.BC-ABC}{r} - M, \text{ and the six remaining ones = } R+W) \text{ it} \\ & \text{will be = } \frac{S.r-1.BC-ABC}{r} + R+W-M. \end{aligned}$$

When A is the oldest of the three lives, the same general rule may be obtained. In this case, by exchanging the symbols,

$$\begin{aligned} & \text{the different series will become = } \frac{S}{6abc} \times \frac{abc'}{r} + \frac{ms.c'+c''}{r^2} + \&c. - \\ & \frac{S}{6abc} \times \frac{bs.c'}{r} + \frac{mt.c'+c''}{r^2} + \&c. + \frac{S}{3abc} \times \frac{am.a'}{r} + \frac{ns.c'+c''}{r^2} + \&c. + \\ & \frac{2S}{3abc} \times \frac{ms.c'}{r} + \frac{nt.c'+c''}{r^2} + \&c. + \frac{S}{c} \times \frac{c'}{r} + \frac{c'}{r^2} + \frac{c''}{r^3} + \&c. - \end{aligned}$$

$$\begin{aligned}
& \frac{S}{bc} \times \frac{mc'}{r} + \frac{n \cdot c' + c''}{r^2} + \&c. - \frac{S}{2ac} \times \frac{ac'}{r} + \frac{sc''}{r^2} + \frac{tc'''}{r^3} + \&c. - \\
& \frac{S}{2ac} \times \frac{sc'}{r} + \frac{t \cdot c''}{r^2} + \frac{uc'''}{r^3} + \&c. - \frac{2S}{3 \cdot bcr} \times \frac{ms \cdot c'}{r} + \frac{nt \cdot c' + c''}{r^2} + \&c. - \\
& \frac{S}{3 \cdot bcr} \times \frac{ms \cdot c'}{r} + \frac{nu \cdot c' + c''}{r^2} + \&c. + \frac{S}{6abc} \times \frac{sc'}{r} + \frac{ot \cdot c' + c''}{r^2} + \&c. - \\
& \frac{S}{6abc} \times \frac{nt \cdot c'}{r} + \frac{cu \cdot c' + c''}{r^2} + \&c. + \frac{S}{bcr} \times \frac{mc'}{r} + \frac{n \cdot c' + c''}{r^2} + \&c. = S \text{ into} \\
& \frac{2 \cdot r-1 \cdot \overline{AB-ABC}}{3r} + \frac{\beta}{6b} \times \frac{\alpha \cdot \overline{HF-HFC}}{a} - \frac{\overline{AF-AFC}}{3a} + \frac{\alpha \cdot \overline{HB-HBC}}{3a} \\
& - \frac{s}{3ar} \times \overline{NB-NBC} + \frac{m \cdot \overline{PN-PNC}}{2b} + \frac{m \cdot \overline{AP-APC}}{6br} + \frac{r-1 \cdot \overline{V-C}}{r} - \\
& \frac{r-1 \cdot \overline{B-BC}}{r} - \frac{r-1 \cdot \overline{A-AC}}{2r} - \frac{\alpha \cdot \overline{H-HC}}{2a} + \frac{s \cdot \overline{N-NC}}{2ar}. \text{ In order to}
\end{aligned}$$

get the same general rule as that given above in the case of B or C's being the oldest of the three lives, it is to be observed, that the first five expressions, by the second part of the solution

of Prob. 1. appear to be $= \frac{r-1 \cdot \overline{AB-ABC}}{r} - M + E$. And (sup-

posing Γ to represent the value of S on the contingency of A's dying after C) that the last three expressions appear to be

$= -\Gamma - \frac{r-1 \cdot \overline{A-AC}}{r}$. But E is $= R + \frac{r-1 \cdot \overline{B-AB}}{r}$, and W must

be $= \frac{r-1 \cdot \overline{V-A-C+AC}}{r} - \Gamma$; therefore, the sum of the above

expressions may be easily found $= \frac{S \cdot r-1 \cdot \overline{BC-ABC}}{r} + R + W - M$.

If the three lives be of equal age, the value of the reversion will be $= \frac{2S \cdot r-1}{3r} \times \overline{V-L}$. This expression may be derived either from the foregoing general rule, or immediately from

from the different series, and is known to be accurately true from self-evident principles. Q. E. D.

I have now given general rules for determining the values of reversionions depending upon three lives in every case which, as far as I can discover, will admit of an *exact* solution. The remaining cases, which are nearly equal in number to those I have investigated, involve a contingency for which it appears very difficult to find such a general expression as shall not render the rules much too complicated and laborious. The contingency to which I refer is that of *one life's failing after another in any given time*. The fractions expressing this probability are every year increasing, so that the value of the reversion must be represented by as many series at least as are equal to the difference between the age of one of the lives, and that of the oldest life in the table of observations. I have indeed so far succeeded in the method of approximation as that the reversion may be generally ascertained within about $\frac{1}{50}$ th part of its exact value; but I shall not trouble the Royal Society at present with these investigations.

The 34th, 35th, and 36th problems in Mr. SIMPSON's Select Exercises involve this contingency, and, by the assistance of M. DE MOIVRE's hypothesis, admit of an easy solution. But such is the fallacy of this hypothesis, that it renders Mr. SIMPSON's conclusions obviously wrong, though his reasoning is perfectly correct: for it cannot surely be an *equal* chance in *all cases* that one life shall die after another. In the short term of a single year the chances are indeed so nearly equal, that it would be wrong to perplex the solution by attempting

greater accuracy. But when the number of years, and the difference between the ages of the two lives are, considerable, those chances must vary in proportion; and, therefore, unless the contingency is blended with another which shall very much diminish the probability of the event, the solution, by thus indiscriminately supposing the chances to be equal, must be rendered extremely inaccurate. In Mr. SIMPSON's 36th problem the solution by this means appears to be absurd: for, in the particular case in which C is the oldest of the three lives, the value of the reversionary annuity becomes $= \frac{C-AC}{2}$; that is, the value of an annuity in this case during the life of C after B and A, provided A dies first, is the same whatever be the age of B; for no mention is made of his life in the foregoing expression. It should be observed, however, that the rule itself is strictly true, and that the error arises from Mr. SIMPSON's having been misled by the hypothesis in determining the probability of B's dying after A in his investigation of the 34th problem, which is applied to the solution of this problem*.

I have declined giving specimens of the different values of the reversions as deduced from the foregoing rules and those which have been hitherto published, not only from an apprehension of becoming tedious, but also from the conviction that at present they are unnecessary; those which I have formerly given being, I think, sufficient to prove the inaccuracy of M. DE MOIVRE's hypothesis. In those instances in which I have compared some of the foregoing rules with the approxi-

* It is proper to observe, that I have followed Mr. SIMPSON's method of determining this contingency in the 23d, 27th, 28th, and 29th Problems in my Treatise on Annuities.

mations now in use, I have invariably found the latter to be erroneous; nay, in some cases, the values were almost twice as great as they ought to have been. This is particularly true when one of the lives is very young, and both or either of the other lives are very old. In reversions of this kind I believe that this is generally the case, and that it seldom happens that the ages of the three lives are nearly equal. The approximations therefore can hardly ever be used with safety, and it will certainly be most prudent not to have recourse to them when the correct values can be obtained. Should the difficulties attending the solution of the remaining problems which involve three lives be surmounted (and the task may not perhaps be impossible), the hypothesis of an equal decrement of life, as far as it relates to any useful purpose in the doctrine of annuities, may then be totally abandoned. Or should it even be found impracticable to deduce solutions of those problems which are strictly and accurately true; yet, I am satisfied from my own experience that such near approximations may be procured as to render this hypothesis equally unnecessary.

